EQUATION OF DYNAMICAL THERMOELASTICITY IN A MEDIUM WITH THERMAL MEMORY

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We derive a linearized system of dynamical thermoelasticity equations for an isotropic medium with thermal memory. We prove a uniqueness theorem and reciprocity theorem for the corresponding boundary-value problem.

Under conditions of intense heating at low temperatures, experiments show that thermal excitations propagate with a finite velocity in the form of waves: the so-called second sound [1-4]. Analysis of thermally stressed states under these conditions is usually done in the framework of generalized thermomechanics [5], based on the hyperbolic heat-conduction equation. Recently a nonlinear theory of thermoviscoelasticity including thermal memory has appeared [6], where by memory we mean that the entire past history of the thermal quantities affects the present state of the material. This theory is based on thermodynamics and gives a natural description from the physical point of view of the thermoelastic state in cases where the propagation of second sound is possible. In [6] it was assumed that the behavior of the material at each point is characterized by four response functionals: $\hat{\psi}$, $\hat{\sigma}_{ij}$, $\hat{\eta}$ and $\hat{q}_i$. The present value of the free energy $\psi = \psi(X_i, t)$, the stress $\sigma_{ij} = \sigma_{ij}(X_i, t)$, the entropy $\eta = \eta(X_i, t)$, and the heat flux $q_i = q_i(X_i, t)$ are determined in terms of these functionals and the thermal history of the material point up to the present time:

$$\psi(X, t)=\hat{\psi}(\Lambda^t); \quad \sigma_{ij}(X, t)=\hat{\sigma_{ij}}(\Lambda^t); \quad \eta(X, t)=\hat{\eta}(\Lambda^t); \quad q_i(X, t)=\hat{q_i}(\Lambda^t), \tag{1}$$

where we use the notation

$$\Lambda^t \equiv (\varepsilon_{ij}, T, \overline{T}^t, \overline{g}_i^t). \tag{2}$$

The history and total history of a function f up to the present time t are denoted by f^t and $\bar{f}^t$, respectively:

$$f^t(s)=f(t-s); \quad \bar{f}^t(s)=\int_0^s f^t(\lambda)\,d\lambda. \tag{3}$$

From the nonlinear functional equations (1) one can find the propagation velocities and damping factors of thermoelastic waves. But the magnitude of the discontinuity on the wave front and the temperature and stress distributions cannot be obtained in this way. An analysis of a thermostressed state with thermal memory can be carried out by linearizing (1).

Let the temperature of the body be T_0 in the undeformed and unstressed state. We will consider the case of small deformations and changes in temperature:

$$\left|\frac{T-T_0}{T_0}\right| \ll 1; \quad \varepsilon_{ij}(t)=\sup_t |u_{i,j}(t)| \ll 1. \tag{4}$$

Then we assume that the pair of functions $[T(\cdot), g_i(\cdot)]$ is close to the temperature of the natural state $T_0 > 0$ in the following sense:

$$\sup_{-\infty<t<t_0} \{|T(t)-T_0|+|g_i(t)|\} < \delta_0, \tag{5}$$

where δ_0 is an infinitesimal quantity.

We introduce the concept of an equilibrium thermal history of the natural state

$$\Lambda_0 \equiv (O_{ij}^c, T_0, T_0 s, O_i^c), \tag{6}$$

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where 0^c_{ij}, 0^c_i are constant tensor and vector functions with the values 0_{ij} and 0_i, respectively. Expanding the response functionals in series about the equilibrium thermal history, we write:

$$q_i(X_i,\ t) = \hat{q}_i(\Lambda_0) + D_{\varepsilon_{ij}}\hat{q}_i(\Lambda_0)\,\varepsilon_{ij} + D_T\hat{q}_i(\Lambda_0)(T-T_0) + \delta_{\bar{T}^t}\hat{q}_i(\Lambda_0)(\bar{T}^t - T_0 s) + \delta_{\bar{g}_i^t}\hat{q}_i(\Lambda_0)\,\bar{g}_i^t + o(\delta_0), \quad (7)$$

$$e(X_i,\ t) = \hat{e}(\Lambda_0) + D_{\varepsilon_{ij}}\hat{e}(\Lambda_0)\,\varepsilon_{ij} + D_T\hat{e}(\Lambda_0)(T-T_0) + \delta_{\bar{T}^t}\hat{e}(\Lambda_0)(\bar{T}^t - T_0 s) + \delta_{\bar{g}_i^t}\hat{e}(\Lambda_0)\,\bar{g}_i^t + o(\delta_0), \quad (8)$$

$$\sigma_{ij}(X_i t) = \hat{\sigma}_{ij}(\Lambda_0) + D_{\varepsilon_{ij}}\hat{\sigma}_{ij}(\Lambda_0)\,\varepsilon_{ij} + D_T\hat{\sigma}_{ij}(\Lambda_0)(T-T_0) + \delta_{\bar{T}^t}\hat{\sigma}_{ij}(\Lambda_0)(\bar{T}^t - T_0 s) + \delta_{\bar{g}_i^t}\hat{\sigma}_{ij}(\Lambda_0)(\bar{g}_i^t) + o(\delta_0). \quad (9)$$

In (7)-(9) the symbol D denotes the partial derivative of the response functional with respect to temperature and the deformation; δ is the partial derivative with respect to the temperature and temperature gradient history.

An isotropic material has a center of symmetry and therefore one has the relations

$$D_T\hat{q}_i(\Lambda_0) = 0; \quad \delta_{\bar{T}^t}\hat{q}_i(\Lambda_0) = 0; \quad \delta_{\bar{g}_i^t}\hat{e}(\Lambda_0) = 0; \quad D_{\varepsilon_{ij}}\hat{q}_i(\Lambda_0) = 0;$$

$$\delta_{\bar{g}_i^t}\hat{\sigma}_{ij}(\Lambda_0) = 0. \quad (10)$$

In equilibrium, the stress and heat flux vanish:

$$\hat{q}_i(\Lambda_0) = 0, \quad \hat{\sigma}_{ij}(\Lambda_0) = 0. \quad (11)$$

Ignoring terms of order δ in (7)-(9) and transforming the derivatives with the help of the Rice [7] representation, we find

$$q_i(X_i,\ t) = -\int_0^\infty \alpha(s)\, g_i(t-s)\, ds, \quad (12)$$

$$e(X_i,\ t) = e_0 + c_v(T-T_0) + \varkappa_1\varepsilon_{kk} + \int_0^\infty \beta(s)(T^t - T_0 s)\, ds, \quad (13)$$

$$\sigma_{ij}(X_i,\ t) = 2\varkappa_3\varepsilon_{ij} + \left\{\varkappa_4\varepsilon_{kk} - \varkappa_2(T-T_0) + \int_0^\infty \gamma(s)(T^t - T_0 s)\, ds\right\}\delta_{ij}. \quad (14)$$

Here $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are the relaxation functions of the heat flux, internal energy, and thermal stress. The coefficients are given by

$$\varkappa_1\delta_{ij} = D_{\varepsilon_{ij}}\hat{e}(\Lambda_0); \quad -\varkappa_2\delta_{ij} = D_T\hat{\sigma}_{ij}(\Lambda_0); \quad \varkappa_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \varkappa_4\delta_{ij}\delta_{kl} = D_{\varepsilon_{ij}}\hat{\sigma}_{ij}(\Lambda_0);$$

$$c_v = D_T\hat{e}(\Lambda_0); \quad e_0 = \hat{e}(\Lambda_0). \quad (15)$$

Finally, using the balance equation for the energy, in which we neglect the dissipative term as quadratic in ε_{ij},

$$\dot{e} = -q_{i,i} + w, \quad (16)$$

and the equation of motion

$$\rho\ddot{u}_i = \sigma_{ij,j} + b_i, \quad (17)$$

we obtain a system of linearized dynamical thermoelasticity equations for an isotropic medium with thermal memory in the form:

$$\rho\ddot{u}_i + \varkappa_2\vartheta_{,i} - \int_0^\infty \gamma(s)\,\vartheta_{,i}(t-s)\, ds = \varkappa_3 u_{i,jj} + (\varkappa_4 + \varkappa_3)\, u_{j,ji} + b_i, \quad (18)$$

$$c_v\ddot{\vartheta} + \beta(0)\,\dot{\vartheta} + \int_0^\infty \beta'(s)\,\dot{\vartheta}(t-s)\, ds + \varkappa_1\ddot{\varepsilon}_{kk} = \alpha(0)\,\vartheta_{,ii} + \int_0^\infty \alpha'(s)\,\vartheta_{,ii}(t-s)\, ds + \dot{w}, \quad (19)$$

$$\sigma_{ij} = 2\varkappa_3\varepsilon_{ij} + \left[\varkappa_4\varepsilon_{kk} - \varkappa_2\vartheta + \int_0^\infty \gamma(s)\,\vartheta(t-s)\, ds\right]\delta_{ij}, \quad (20)$$

where $\vartheta = T - T_0$ and δ_{ij} is the Kronecker delta.

We prove the uniqueness of the solution of (18)-(20). The initial conditions and boundary conditions have the form

$$\vartheta(t) = \dot{\vartheta}(t) = u_i(t) = \dot{u}_i(t) = \sigma_{ij}(t) = \dot{\sigma}_{ij}(t) = 0, \; -\infty < t \leqslant 0, \quad (21)$$

$$\sigma_{ij}(P,\ t)\, n_j = p_i(P,\ t); \quad u_i(P,\ t) = \Psi_i(P,\ t); \quad P \in S, \quad (22)$$

$$\vartheta(P,\ t) = \theta(P,\ t); \quad P \in S_1; \quad (23)$$

$$\int_0^\infty \alpha(s)\,\vartheta_{,i}(P,\ t-s)\,n_i ds = 0, \quad P \in S_2, \tag{24}$$

where S_1 is the part of the surface over which the temperature is specified and S_2 is the remaining portion of the surface which is assumed to be perfectly thermally insulated.

We assume that the following relations are satisfied:

$$\alpha(0) > 0, \quad \beta(0) > 0, \quad \gamma(0) \leqslant 0, \quad \rho,\ \varkappa_1,\ \varkappa_2,\ \varkappa_3,\ \varkappa_4,\ c_v \geqslant 0. \tag{25}$$

Let two different solutions of the problem be given by $u_i^{(1)}$, $\varepsilon_{ij}^{(1)}$, $\sigma_{ij}^{(1)}$, $\vartheta^{(1)}$, and $u_i^{(2)}$, $\varepsilon_{ij}^{(2)}$, $\sigma_{ij}^{(2)}$, $\vartheta^{(2)}$. Then $u_i^{(0)} = u_i^{(1)} - u_i^{(2)}$, $\sigma_{ij}^{(0)} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$, $\vartheta^{(0)} = \vartheta^{(1)} - \vartheta^{(2)}$ are also solutions of the problem for homogeneous boundary conditions. We apply the Laplace transform to (18), (19), (20) and the boundary conditions (22), (23), and (24), using the initial condition (21). With the use of (18), (20) and boundary conditions (22)-(24) we obtain the following relation:

$$\int_V [\rho p^2 \bar{u}_i^{(0)} \bar{u}_i^{(0)} + 2\varkappa_3 \bar{\varepsilon}_{ij}^{(0)} \bar{\varepsilon}_{ij}^{(0)} + \varkappa_4 \bar{\varepsilon}_{kk}^{(0)} \bar{\varepsilon}_{kk}^{(0)} - (\varkappa_2 - \bar{\gamma})\,\bar{\vartheta}^{(0)} \bar{\varepsilon}_{kk}^{(0)}]\, dV = 0. \tag{26}$$

Eliminating the last term in (26) with the help of the Laplace transform of the heat equation (19) and the identity

$$\int_S \bar{\alpha}\,\bar{\vartheta}^{(0)} \bar{\vartheta}_{,i}^{(0)} n_i ds = 0, \tag{27}$$

we obtain the expression:

$$\int_V \{\rho p^2 \bar{u}_i^{(0)} \bar{u}_i^{(0)} + 2\varkappa_3 \bar{\varepsilon}_{ij}^{(0)} \bar{\varepsilon}_{ij}^{(0)} + \varkappa_4 \bar{\varepsilon}_{kk}^{(0)} \bar{\varepsilon}_{kk}^{(0)} + \frac{(\varkappa_2 - \bar{\gamma})}{\varkappa_1 p}\,[p(c_v + \bar{\beta})\,\bar{\vartheta}^{(0)} \bar{\vartheta}^{(0)} + \bar{\alpha}\bar{\vartheta}_{,i}^{(0)} \bar{\vartheta}_{,i}^{(0)}]\}\, dV = 0. \tag{28}$$

We consider real values of the Laplace transform parameter p such that $p > p_0$, where p_0 is the farthest singularity of the solution from the origin on the right half of the real axis. From the operational relation

$$\lim_{p\to\infty} p\bar{f}(p) = f(0) \tag{29}$$

and (25) it follows that

$$p\bar{\alpha}(p) > 0; \quad p\bar{\beta}(p) > 0; \quad p\bar{\gamma}(p) \leqslant 0 \quad \text{for} \quad p \to \infty. \tag{30}$$

Therefore, for large enough p, Eq. (28) consists of nonnegative terms and can only be satisfied if

$$\bar{u}_i^{(0)} = 0, \quad \bar{\vartheta}^{(0)} = 0 \quad \text{for} \quad p > p_0. \tag{31}$$

Thus, it follows that $u_i^{(0)} = 0$ and $\vartheta^{(0)} = 0$ [8]. This means that $u_i^{(1)} = u_i^{(2)}$, $\vartheta^{(1)} = \vartheta^{(2)}$, $\sigma_{ij}^{(1)} = \sigma_{ij}^{(2)}$ and the solution of the thermoelastic boundary-value problem for an isotropic medium with thermal memory is unique.

Let a heat source w and body force b_i act on an isotropic elastic body with thermal memory, and let the stress p_i and temperature θ be given on its surface. We use the abbreviation $I \equiv \{b_i,\ p_i,\ w,\ \theta\}$ for the sources; the corresponding responses (displacement and temperature) are abbreviated by the symbol $C \equiv \{u_i,\ \vartheta\}$. A second set of sources and responses is denoted by $I' \equiv \{b_i',\ p_i',\ w',\ \theta'\}$ and $C' \equiv \{u_i'\ \vartheta'\}$. Taking the Laplace transform of (18)-(20), we obtain the identities

$$\bar{\sigma}_{ij} \bar{u}'_{i,j} - \bar{\sigma}'_{ij} \bar{u}_{i,j} = (\varkappa_2 - \bar{\gamma})\,(\bar{\vartheta}' \bar{\varepsilon}_{kk} - \bar{\vartheta}\,\bar{\varepsilon}'_{kk}), \tag{32}$$

$$\bar{\alpha}\,(\bar{\vartheta}_{,ii} \bar{\vartheta}' - \bar{\vartheta}'_{,ii} \bar{\vartheta}) = \varkappa_1 p\,(\bar{\varepsilon}_{kk} \bar{\vartheta}' - \bar{\varepsilon}'_{kk} \bar{\vartheta}) - (\bar{w}\,\bar{\vartheta}' - \bar{w}' \bar{\vartheta}). \tag{33}$$

Integrating (32) and (33) with respect to volume and transforming with the help of the Ostrogradskii–Gauss theorem, the equation of motion, and the boundary conditions, after elimination of a common term we obtain the equation

$$\varkappa_1 p \Big[\int_S (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i)\, ds + \int_V (\bar{b}_i \bar{u}'_i - \bar{b}'_i \bar{u}_i)\, dV\Big] = (\varkappa_2 - \bar{\gamma})\,\Big[\int_S \bar{\alpha}\,(\bar{\theta}' \bar{\vartheta}_{,i} - \bar{\theta}\,\bar{\vartheta}'_{,i})\, ds - \int_V \bar{w}\,\bar{\vartheta}' - \bar{w}' \bar{\vartheta})\, dV\Big]. \tag{34}$$

Applying the inverse Laplace transform and the convolution theorem to (34) we obtain the reciprocity theorem:

$$\varkappa_1 \int\limits_S dS \int\limits_0^t \left[p_i(X_i,\ t-\tau)\frac{\partial u_i'(X_i,\ \tau)}{\partial\tau} - p_i'(X_i,\ t-\tau)\frac{\partial u_i(X_i,\ \tau)}{\partial\tau}\right] d\tau +$$

$$+ \varkappa_1 \int\limits_V dV \int\limits_0^t \left[b_i(X_i,\ t-\tau)\frac{\partial u_i'(X_i,\ \tau)}{\partial\tau} - b_i'(X_i,\ t-\tau)\frac{\partial u_i(X_i,\ \tau)}{\partial\tau}\right] d\tau =$$

$$= \varkappa_2 \int\limits_V dV \int\limits_0^t [\vartheta'(X_i,\ t-\tau)\, w(X_i,\ \tau) - \vartheta(X_i,\ t-\tau)\, w'(X_i,\ \tau)]\, d\tau - \int\limits_V dV \int\limits_0^t \gamma(t-\tau)\left[\int\limits_0^\tau [\vartheta'(X_i,\ \tau-\xi)\, w(X_i,\ \xi) -\right.$$

$$\left. - \vartheta(X_i,\ \tau-\xi)\, w'(X_i,\ \xi)]\, d\xi\right] d\tau + \varkappa_2 \int\limits_S dS \int\limits_0^t \alpha(t-\tau)\left[\int\limits_0^\tau \theta'(X_i,\ \tau-\xi)\, \times\right.$$

$$\left. \times\, \vartheta_{,i}(X_i,\ \xi) - \theta(X_i,\ \tau-\xi)\, \vartheta'_{,i}(X_i,\ \xi)\, d\xi\right] d\tau - \int\limits_S dS \int\limits_0^t \gamma(t-\tau)\times$$

$$\times \left\{\int\limits_0^\tau \alpha(\tau-\xi)\left[\int\limits_0^\xi [\theta'(X_i,\ \xi-\zeta)\, \vartheta_{,i}(X_i,\ \zeta) - \theta(X_i,\ \xi-\zeta)\, \vartheta'_{,i}(X_i,\ \zeta)]\, d\zeta\right] d\xi\right\} d\tau. \qquad (35)$$

When the medium is infinite, the reciprocity theorem simplifies

$$\varkappa_1 \int\limits_V dV \int\limits_0^t \left[b_i(X_i,\ t-\tau)\frac{\partial u_i'(X_i,\ \tau)}{\partial\tau} - b_i'(X_i,\ t-\tau)\frac{\partial u_i(X_i,\ \tau)}{\partial\tau}\right] d\tau =$$

$$= \varkappa_2 \int\limits_V dV \int\limits_0^t [\vartheta'(X_i,\ t-\tau)\, w(X_i,\ \tau) - \vartheta(X_i,\ t-\tau)\, w'(X_i,\ \tau)]\, d\tau -$$

$$- \int\limits_V dV \int\limits_0^t \gamma(t-\tau)\left[\int\limits_0^\tau \vartheta'(X_i,\ \tau-\xi)\, w(X_i,\ \xi) - \vartheta(X_i,\ \tau-\xi)\, w'(X_i,\ \xi)\, d\xi\right] d\tau. \qquad (36)$$

If in (35) and (36) we take the relaxation function for the heat flux in the form of the Maxwell–Kattaneo function

$$\alpha(t) = \frac{\lambda}{\tau_r}\exp\left(-\frac{t}{\tau_r}\right)$$

and assume there is no relaxation of the stress or internal energy, and express the linearization coefficients in terms of the Lamé coefficients

$$\varkappa_1 = \alpha_t(3\lambda + 2\mu)\, T_0;\quad \varkappa_2 = \alpha_t(3\lambda + 2\mu);\quad \varkappa_3 = \mu;\quad \varkappa_4 = \lambda,$$

then we obtain the well-known results of generalized thermomechanics [5].

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